UNSTEADY HYPERSONIC FLOW OVER A CONE WITH A BLUNTED TIP

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UNSTEADY SUPERSONIC FLOW OVER A CONE WITH A BLUNTED TIP

(Nestatsionarnoe Sverkhzvukovoe Obtekanie Konusa s Okruglennoi Vershinoi)

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§1. Consider the supersonic flow over a "sphere-cone" body (ABC in Fig. 1), performing plane angular oscillations relative to the center O according to the law

$$\alpha = \alpha_0 \cos \omega t \tag{1.1}$$

where α is the instantaneous value of the angle of attack. In Fig. 1 ME is a condensation shock and QP is the sonic line. We will assume that the conditions

$$\alpha_0 << 1 \qquad \frac{\omega L}{V_1} << 1 \tag{1.2}$$

are satisfied, where V_1 is the free stream velocity and L is a characteristic length of the body. Therefore, the perturbations caused by the oscillating body will be small so that the problem can be solved by the method of small disturbances. We note that the second condition (1.2) is satisfied to a high degree of accuracy for supersonic flight.

Restricting ourselves to the linear approximation with respect to the frequency, we will neglect quantities of order α_0^2 and order $\alpha_0\omega^2$. Then the parameters of the gas: the velocity V, the pressure p and the density ρ can be represented in the following form:

$$\vec{v} = \vec{v}_0 + \alpha\vec{v}_\alpha + \dot{\alpha}\vec{v}_{\dot{\alpha}}\,, \qquad p = p_0 + \alpha p_\alpha + \dot{\alpha} p_{\dot{\alpha}}\,, \qquad \rho = \rho_0 + \alpha\rho_\alpha + \dot{\alpha}\rho_{\dot{\alpha}} \tag{1.3}$$

The parameters with index 0 describe the basic field arising from the stationary flow over the body. Parameters with indices α and $\dot{\alpha}$ describe the perturbation field which occurs in phase with the angle of attack and the angular velocity respectively and do not depend on the time and the frequency.

Let us substitute the expansions (1.3) into the system of equations of gas dynamics written in a moving coordinate system connected to the body. Then, for determining parameters with index α we obtain the system of linear equations

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$$\operatorname{grad}(\vec{v}_0\vec{v}_\alpha) - \vec{v}_0 \times \operatorname{rot} \vec{v}_\alpha - \vec{v}_\alpha \operatorname{rot} \vec{v}_0 = \frac{\rho_\alpha}{\rho_0^2} \operatorname{grad} p_0 - \frac{1}{\rho_0} \operatorname{grad} p_\alpha$$

$$\operatorname{div}(\rho_0\vec{v}_\alpha + \rho_\alpha\vec{v}_0) = 0 \tag{1.4}$$

$$\frac{p_0}{\rho_0^\gamma} \vec{v}_0 \operatorname{grad} \left(\frac{p_\alpha}{p_0} - \gamma \frac{\rho_\alpha}{\rho_0}\right) + \vec{v}_\alpha \operatorname{grad} \left(\frac{p_0}{\rho_0^\gamma}\right) = 0$$

and for determining parameters with index $\dot{\alpha}$

$$\vec{v}_\alpha + \operatorname{grad}\left[\vec{v}_0(\vec{v}_{\dot{\alpha}} - \vec{v}_{e\dot{\alpha}})\right] - \vec{v}_0 \times \operatorname{rot} \vec{v}_{\dot{\alpha}}$$

$$- (\vec{v}_{\dot{\alpha}} - \vec{v}_{e\dot{\alpha}}) \times \operatorname{rot} \vec{v}_0 = \frac{\rho_{\dot{\alpha}}}{\rho_0^2} \operatorname{grad} p_0 - \frac{1}{\rho_0} \operatorname{grad} p_{\dot{\alpha}} \tag{1.5}$$

$$\rho_\alpha + \operatorname{div}(\rho_{\dot{\alpha}}\vec{v}_0 + \rho_0\vec{v}_{\dot{\alpha}}) - \vec{v}_{e\dot{\alpha}} \operatorname{grad} \rho_0 = 0$$

$$\frac{p_0}{\rho_0^\gamma}\left[\left(\frac{p_\alpha}{p_0} - \gamma \frac{\rho_\alpha}{\rho_0}\right) + \vec{v}_0 \operatorname{grad} \left(\frac{p_{\dot{\alpha}}}{p_0} - \gamma \frac{\rho_{\dot{\alpha}}}{\rho_0}\right)\right] + (\vec{v}_{\dot{\alpha}} - \vec{v}_{e\dot{\alpha}}) \operatorname{grad} \frac{p_0}{\rho_0^\gamma} = 0$$

where v_e is the moving velocity vector. Here the parameters with index 0 satisfy the system of equations

$$\operatorname{grad} \frac{v_0^2}{2} - \vec{v}_0 \times \operatorname{rot} \vec{v}_0 = - \frac{1}{\rho_0} \operatorname{grad} p_0$$

$$\tag{1.6}$$

$$\operatorname{div} \rho_0\vec{v}_0 = 0 \qquad \vec{v}_0 \operatorname{grad} \frac{p_0}{\rho_0^\gamma} = 0$$

which describe a stationary gas flow.

We will suppose that the spherical and conical parts of the surface are joined without jumps of the slope generatrices (without corners). Furthermore, let the semiangle of the cone, θ_s, be such that the point of junction B is always found in the supersonic region. The first family characteristic BD, starting from the point of intersection B, divides the region between the condensation shock and the surface of the body into two parts for the stationary flow over the body at zero angle of attack. In the region ABDM the flow is the same as the flow over an isolated sphere and does not depend upon the presence of the conical part. In the region DBC the flow is determined by the motion of the conical part of the surface of the body and by the propagation of the initial perturbations, given on the characteristic BD, and also by the stipulated flow over the spherical part of the body.

§2. We will begin with the determination of the flow in region ABDM. For this let us consider the flow over the sphere performing plane angular oscillations with respect to the center 0 (see Fig. 2) according to the law (1.1). The motion of the sphere can be decomposed into a translational motion with velocity $\vec{v}_e^{\,*}$ equal to the absolute velocity of the center, 0_1, of the sphere and a rotational motion around the center with velocity $\vec{v}_e^{\,**}$, determined by the law (1.1). We will introduce, in the examination, a spherical system of coordinates $R_2, \tilde{\theta}_2, \mu_2$ connected with the body with the relative velocity

$$\vec{v}_e = \vec{v}_e^{\,*} + \vec{v}_e^{\,**}, \qquad (2.1)$$

and a partially connected system of coordinates (spherical coordinates $R_1, \tilde{\theta}_1, \mu_1$ and rectangular coordinates x, y, z), translating with velocity v_e^* (Fig. 2).

Let us consider the perturbations coinciding in phase, as is defined, with the angle of attack. In the linear approximation with respect to frequency these perturbations correspond to the stationary flow over a body at an angle of attack α. It is evident that for the stationary flow over the sphere the gas dynamic parameters in the system of coordinates $R_1, \tilde{\theta}_1, \mu_1$ do not depend on the angle of attack but in the system of coordinates $R_2, \tilde{\theta}_2, \mu_2$, in the linear approximation with respect to α, we have the relations

$$f(\alpha) = f_0 + \alpha \left(\frac{\partial f}{\partial \alpha}\right)_{\alpha=0} \quad , \quad \vec{v}(\alpha) = \vec{v}_0 + \alpha \left(\frac{\partial \vec{v}}{\partial \alpha}\right)_{\alpha=0}$$

where f is an arbitrary scalar parameter. Making use of the relations for the partial derivatives with respect to time in the moving system of coordinates, we obtain for the scalar parameter f

$$f_\alpha = \left(\frac{\partial f}{\partial \alpha}\right)_{\alpha=0} = \vec{v}^{**}_{e\dot{\alpha}} \text{ grad } f_0 \tag{2.2}$$

and for the vector velocity $\vec{v}$

$$\vec{v}_\alpha = \left(\frac{\partial \vec{v}}{\partial \alpha}\right)_{\alpha=0} = (v^{**}_{e\dot{\alpha}} \nabla)\vec{v}_0 - \left[\vec{k} \times \vec{v}_0\right] \tag{2.3}$$

or

$$\vec{v}_\alpha = \text{grad}(\vec{v}^{**}_{e\dot{\alpha}} \vec{v}_0) - \vec{v}^{**}_{e\dot{\alpha}} \times \text{rot } \vec{v}_0$$

where $\vec{k}$ is a unit vector directed along the axis of the rectangular system of coordinates, and $\vec{v}^{**}_{e\dot{\alpha}} = \vec{v}^{**}_e/\dot{\alpha}$.

Relations (2.2), (2.3) permit us to express the parameters with indices α by means of the parameters of the stationary flow over the sphere.

Let us now consider the perturbations coinciding in phase, as defined, with the angular velocity. The rotational motion of the sphere around its center O_1 does not cause perturbations in the flow of an ideal gas. Therefore, the perturbations with indices $\dot{\alpha}$ are determined by the translational flow over the sphere with relative velocity

$$\vec{v}^{*}_e = - \dot{\alpha} \ell_1 \vec{j} \tag{2.4}$$

where $\vec{j}$ is a unit vector directed along the y axis of the rectangular coordinate system and ℓ_1 is the distance between the points O_1 and O, measured in the direction of the axis $\theta_2 = 0$ (see Fig. 2). In the linear approximation with respect to frequency the translational motion of the body, with velocity (2.4) in the partially fixed system of coordinates, is equivalent to a stationary flow at a fictitious angle of attack

$$\alpha_\Phi = - \frac{\dot{\alpha}\ell_1}{V_1} \tag{2.5}$$

Taking this into account, we obtain for the arbitrary scalar parameter f

$$f_{\dot{\alpha}} = - \frac{\ell_1}{V_1} f_\alpha \tag{2.6}$$

and for the velocity vector $\vec{v}$

$$\vec{v}_{\dot{\alpha}} - \vec{v}^{\,*}_{e\dot{\alpha}} = - \frac{\ell_1}{V_1} \vec{v}_\alpha \,, \quad \text{or} \quad \vec{v}_{\dot{\alpha}} - \vec{v}_{e\dot{\alpha}} = - \frac{\ell_1}{V_1} \vec{v}_\alpha - \vec{v}^{\,**}_{e\dot{\alpha}} \tag{2.7}$$

Relations (2.6) and (2.7) also permit the parameters with indices $\dot{\alpha}$ to be expressed by means of parameters of the stationary flow over the sphere.

It is easy to show that the obtained solutions (2.3) and (2.6), (2.7) satisfy the system of equations (1.4), (1.5) and the boundary conditions on the body

$$v_{\alpha R} = 0 \,, \quad v_{\dot{\alpha}R} - v_{e\dot{\alpha}R} = 0 \quad \text{for} \quad R = R_s \tag{2.8}$$

where the index R denotes the projection of a vector on the R axis of the spherical system. In fact, because the system of equations (1.4) and the first relation (2.8) can be obtained by variation, with respect to the angle of attack α, of the exact system of differential equations of gas dynamics and the boundary conditions on the surface of the body, written for the case of

stationary flow, then it is clear that the solution (2.2), (2.3) satisfies both the system of equations (1.4) and correspondingly condition (2.8). Substituting (2.2) and the second relation (2.3) into the system of equations (1.5) and using relation (2.1), we obtain

$$\begin{gathered}
\operatorname{grad}\left[\vec{v}_0(\vec{v}_{\dot{\alpha}} - \vec{v}^{\,*}_{e\dot{\alpha}})\right] - \vec{v}_0 \times \operatorname{rot}\,(\vec{v}_{\dot{\alpha}} - \vec{v}^{\,*}_{e\dot{\alpha}}) \\
- (\vec{v}_{\dot{\alpha}} - \vec{v}_{e\dot{\alpha}}) \times \operatorname{rot}\,\vec{v}_0 = \frac{\rho_{\dot{\alpha}}}{\rho_0^2}\operatorname{grad} p_0 - \frac{1}{\rho_0}\operatorname{grad} p_{\dot{\alpha}} \\
\operatorname{div}\left[\rho_0(\vec{v}_{\dot{\alpha}} - \vec{v}^{\,*}_{e\dot{\alpha}}) + \rho_{\dot{\alpha}}\vec{v}_0\right] = 0 \\
\frac{p_0}{\rho_0^{\gamma}}\vec{v}_0 \operatorname{grad}\left(\frac{p_{\dot{\alpha}}}{p_0} - \gamma\frac{\rho_{\dot{\alpha}}}{\rho_0}\right) + (\vec{v}_{\dot{\alpha}} - \overset{*}{v}_{e\alpha})\operatorname{grad}\frac{p_0}{\rho_0^{\gamma}} = 0
\end{gathered} \tag{2.9}$$

Comparing (2.9) with (1.4) and the second relation (2.8) with the first, we see that the solution (2.6), (2.7) satisfies the system of equations (2.5) and correspondingly condition (2.8).

§3. Let us now dwell on the calculation of the flow in the region DBC. We introduce a special cylindrical coordinate system, connected with the body (Fig. 1). The axis of this system coincides with the axis of symmetry of the body. The meridian plane is fixed by the angle $\psi = -\mu_2$, and the position of a point in the meridian plane is fixed by polar coordinates **R**, θ with pole at the point of intersection of the generating spherical and conical parts of the body. The solution of the system of equations (1.6), (1.4) and (1.5) must satisfy the boundary conditions on the surface of the body

$$v_{0\theta} = 0\,, \quad v_{\alpha\theta} = 0\,, \quad v_{\dot{\alpha}\theta} - v_{e\dot{\alpha}\theta} = 0 \quad \text{for} \quad \theta = \theta_c \tag{3.1}$$

where the index θ denotes the projection of a vector on the θ axis, and must take the given values on the characteristic surface

BD, which is determined as a result of the calculation of the flow over the spherical part of the body.

Because the flow in the region DBC is supersonic, then for the solution of the nonlinear system of equations (1.6) and for the solution of the linear system (1.4) and (1.5) we may use the method of characteristics. However, this method for obtaining numerical results requires a great outlay of labor and the application of high speed electronic machines in each concrete case. Therefore, we cite here an approximate analytical method of solution of the problem. A solution will be sought in the form†

$$f = (f_{00} + r f_{01} + \dots) + \alpha(f_{\alpha 0} + r f_{\alpha 1} + \dots)\cos\psi + \dot{\alpha}(f_{\dot{\alpha}0} + r f_{\dot{\alpha}1} + \dots)\cos\psi$$

$$v_\psi = \alpha(v_{\alpha\psi 0} + r v_{\alpha\psi 1} + \dots)\sin\psi + \dot{\alpha}(v_{\dot{\alpha}\psi 0} + r v_{\dot{\alpha}\psi 1} + \dots)\sin\psi \tag{3.2}$$

where f is an arbitrary unknown parameter, excepting the projection of the vector velocity on the ψ axis. The coefficients in the expansion (3.2) depend only upon θ.

Substituting the corresponding expansions into the system of equations (1.6), (1.4) and (1.5), written beforehand in terms of projections on the axis of the cylindrical system of coordinates R, θ, ψ, we obtain systems of ordinary differential equations with respect to θ. Solutions of these systems must satisfy the boundary condition at $\theta = \theta_s$ and the initial conditions at $\theta = \theta^*$, where θ^* is the angle of inclination of the tangent to the characteristic BD at the point B (for $r = 0$). For parameters with index 00 we obtain the following system of nonlinear equations

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†This form of dependence upon ψ is necessary from considerations of the boundary conditions on the body and the form of the initial data on the characteristic BD (see the following).

$$v'_{0r0} = v_{0\theta 0}, \qquad v'_{0\theta 0} + v_{0r0} = -\frac{p'_{00}}{\rho_{00} v_{0\theta 0}}$$

$$\frac{\rho'_{00}}{\rho_{00}} + \frac{v'_{0\theta 0} + v_{0r0}}{v_{0\theta 0}} = 0 \qquad \frac{p_{00}}{\rho_{00}^{\gamma}} = \text{const.} \tag{3.3}$$

and the supplementary final relation

$$\frac{v_{0r0}^2 + v_{0\theta 0}^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p_{00}}{\rho_{00}} = \frac{v_{max}^2}{2} \tag{3.4}$$

where v_{max} is the velocity of the flow into a vacuum and the prime denotes differentiation with respect to θ. Let us substitute the expressions for p'_{00}/ρ_{00}, ρ'_{00}/ρ_{00} and p_{00}/ρ_{00} from equations (3.3) and (3.4) into the result of differentiating (3.4). Taking into account the first equation (3.3), we obtain

$$\left(\frac{\gamma+1}{\gamma-1} v'^{2}_{0r0} + v_{0r0}^2 - v_{max}^2\right)(v''_{0r0} + v_{0r0}) = 0 \tag{3.5}$$

Equating to zero the first factor, with the aid of (3.3) and (3.4), we obtain a Prandtl-Meyer solution. This solution for parameters with index 00 would be required by us in the case of a junction of the spherical and conical sections with a jump in slope at the point B. In the case under consideration it is necessary to equate to 0 the second factor (3.5). Thereby we obtain a solution satisfying the initial and boundary conditions:

$$v_{0r0} = \frac{v^*_{0r0}}{\cos \varphi^*} \cos \varphi, \qquad v_{0\theta 0} = -\frac{v^*_{0r0}}{\cos \varphi^*} \sin \varphi$$

$$p_{00} = p^*_{00}, \qquad \rho_{00} = \rho^*_{00} \qquad (\varphi = \theta - \theta_c) \tag{3.6}$$

where the asterisk superscript denotes values of the parameters at $\theta = \theta^*$. The solution (3.6) describes a uniform translational motion. For the remaining coefficients in the expansion (3.2) we obtain a system of linear differential equations which are easy to integrate in finite form. We will cite the results of the integrations for the coefficients in the expansion of the pressure p, necessary for the calculation of the aerodynamic loads acting on the body

$$p_{01} = \frac{p_{01}^*}{\cos\varphi^*}\cos\varphi, \quad p_{\alpha 0} = p_{\alpha 0}^*, \quad p_{\alpha 1} = \frac{p_{\alpha 1}^*}{\cos\varphi^*}\cos\varphi, \quad p_{\dot\alpha 0} = p_{\dot\alpha 0}^*$$

$$p_{\dot\alpha 1} = \frac{1}{\cos\varphi^*}\left[p_{\dot\alpha 1}^* + 2\rho_{00}^* v_{0r0}^* \tan\varphi^*\right]\cos\varphi - 2\rho_{00}^* \frac{v_{0r0}^*}{\cos\varphi^*}\sin\varphi \tag{3.7}$$

For the completion of the solution of the problem in region DBC, it is necessary to express the initial data on $\theta = \theta^*$ with the aid of the solution obtained in §2 for the flow in region ABDM.

§4. Let us begin with the derivation of the equation of the actual perturbed characteristic surface, passing through the line of intersection of the spherical and conical parts of the surface of the body. We will seek this in the form:

$$\theta = (\theta^* + r\theta_1 + \ldots) + \alpha(\theta_{\alpha 0} + r\theta_{\alpha 1} + \ldots)\cos\psi + \dot\alpha(\theta_{\dot\alpha 0} + r\theta_{\dot\alpha 1} + \ldots)\cos\psi \tag{4.1}$$

The form of the characteristic surface is determined by the condition

$$N - \vec{n}(\vec{v} - \vec{v}_e) = a, \qquad N = \frac{-\partial F/\partial t}{\sqrt{|\mathrm{grad}\, F|^2}}, \qquad \vec{n} = \frac{\mathrm{grad}\, F}{\sqrt{|\mathrm{grad}\, F|^2}} \tag{4.2}$$

where $\vec{n}$ is the unit normal vector, N is the relative velocity of motion of a point of the surface in the direction of the normal,

$F(r,\theta,\psi,t) = 0$ is the equation of the characteristic surface, and a is the local velocity of sound.

Substituting (4.1) into the second and third relations (4.2), we obtain

$$N = - \dot{\alpha}(r\theta_{\alpha 0} + \ldots)\cos\psi$$

$$n_r = - (r\theta_1 + \ldots) - \alpha(r_{\alpha 1} + \ldots)\cos\psi - \dot{\alpha}(r\theta_{\alpha 1} + \ldots)\cos\psi \tag{4.3}$$

$$n_\theta = 1, \quad n_\psi = \alpha\left(r\frac{\theta_{\alpha 0}}{R_0} + \ldots\right)\sin\psi + \dot{\alpha}\left(r\frac{\theta_{\dot{\alpha} 0}}{R_0} + \ldots\right)\sin\psi$$

where $R_0 = BF$ (Fig. 1). Let us substitute the expansions (3.2) and (4.3) into the first relation (4.2). Expanding the functions of θ in a series in terms of $\theta - \theta^*$ by (4.1), we obtain expressions of the coefficients in the expansion of the equation of the characteristic surface (4.1) in terms of the values of the parameters of the gas at $\theta = \theta^*$. We cite the expressions for two of the coefficients which are necessary to us in what follows

$$\theta_{\alpha 0} = - \frac{v^*_{\alpha\theta 0} + a^*_{\alpha 0}}{v'^*_{0\theta 0} + a'^*_{00}}, \qquad \theta_{\dot{\alpha} 0} = - \frac{v^*_{\dot{\alpha}\theta 0} + a^*_{\dot{\alpha} 0}}{v'^*_{0\theta 0} + a'^*_{00}} \tag{4.4}$$

a_{00}, $a_{\alpha 0}$, $a_{\dot{\alpha} 0}$ are the coefficients in the expansion of the local velocity of sound (3.2).

Writing down the condition of continuity of parameters on the actual perturbed characteristic and expanding the functions of θ in a series with respect to $\theta - \theta^*$, we obtain that at $\theta = \theta^*$ it is necessary to satisfy the following relations:

$$\left[f_{00}^{*}\right] = 0, \qquad \left[f_{01}^{*}\right] + \theta_1\left[f_{00}^{'*}\right] = 0$$

$$\left[f_{\alpha 0}^{*}\right] + \theta_{\alpha 0}\left[f_{00}^{'*}\right] = 0$$

$$\left[f_{\alpha 1}\right] + \theta_1\left[f_{\alpha 0}^{'*}\right] + \theta_{\alpha 0}\left[f_{01}^{'*}\right] + \theta_1\theta_{\alpha 0}\left[f_{00}^{''*}\right] + \theta_{\alpha 1}\left[f_{00}^{'*}\right] = 0 \tag{4.5}$$

$$\left[f_{\dot{\alpha} 0}^{*}\right] + \theta_{\dot{\alpha} 0}\left[f_{00}^{'*}\right] = 0$$

$$\left[f_{\dot{\alpha} 1}^{*}\right] + \theta_1\left[f_{\dot{\alpha} 0}^{'*}\right] + \theta_{\dot{\alpha} 0}\left[f_{01}^{'*}\right] + \theta_1\theta_{\dot{\alpha} 0}\left[f_{00}^{''*}\right] + \theta_{\dot{\alpha} 1}\left[f_{00}^{'*}\right] = 0$$

where the bracket denotes a jump in the quantities contained in it from the left to the right for the approximation at $\theta = \theta^*$. The continuity of the parameters of the gas on the actual perturbed characteristic surface and the distinct analytical character of the solution (jumps in derivatives) on both sides of it lead to the fact that for a united solution, describing the flow on the spherical and conical parts of the body it is necessary to take into account discontinuities of parameters along the line $\theta = \theta^*$, as indicated in (4.5).

In the conclusion of this paragraph let us make one remark. If the system of equations (1.6), (1.4) and (1.5) is projected on the axis of the cylindrical system of coordinates and if one eliminates from the parameters of the perturbations (with indices α and $\dot{\alpha}$) the dependence upon ψ, then it is easy to see, that the coefficients for the corresponding derivatives in all three systems coincide, and consequently, the characteristics of the linear systems of equations for the perturbations coincide in the physical plane with the characteristics of the nonlinear systems of equations describing the stationary flow over a body at zero angle of attack. This circumstance is convenient for the integration of the linear system of equations (1.4) and (1.5) by the method of characteristics. However, from the preceding

considerations, it is clear that for calculations with the aid of relations of the type (4.5) it is necessary to take into account discontinuities of the perturbation parameters on the characteristics separating regions with different analytical character of the solutions.

§5. With the results of §2, the coefficient values in the expansion (3.2) are determined on the line $\theta = \theta^*$, by using the approximation on the side of the spherical part of the body. These coefficient values will be denoted by the subscript s.

In the vicinity of the surface $\theta = \theta^*$, $\tilde{\theta}_2 \gg \alpha$, and $\pi - \tilde{\theta}_2 \gg \alpha$, so that $\mu_2 = \mu_1 + O(\alpha)$. Taking this into account, and remembering the expressions

$$v^{**}_{e\dot{\alpha}R} = 0 , \qquad v^{**}_{e\dot{\alpha}\theta} = - R \cos \mu , \qquad v^{**}_{e\alpha\psi} = R \cos \theta \sin \mu \tag{5.1}$$

from (2.3) and (2.2) we find

$$\begin{aligned} f(R_2, \tilde{\theta}_2, \mu_2, \alpha) &= f_0(R_2, \tilde{\theta}_2) - \alpha \frac{\partial f_0}{\partial \tilde{\theta}_2} \cos \mu_2 \\ v_\mu(R_2, \tilde{\theta}_2, \mu_2, \alpha) &= \frac{\alpha v_{0\tilde{\theta}}}{\sin \tilde{\theta}_2} \sin \mu_2 \end{aligned} \tag{5.2}$$

One can easily find the desired values of the coefficients for the expansion (3.2) for $\theta = \theta^*$ on the side of the spherical part with equations (5.2) and (2.6) and using the relations between components of the velocity vector in cylindrical and spherical coordinate systems

$$\begin{aligned} v_r &= - v_R \cos(\theta + \tilde{\theta}_2) + v_\theta \sin(\theta + \tilde{\theta}_2) \\ v_\theta &= v_R \sin(\theta + \tilde{\theta}_2) + v_\theta \cos(\theta + \tilde{\theta}_2) v_\psi = - v_\mu \end{aligned} \tag{5.3}$$

and the relations

$$\left(\frac{\partial R_2}{\partial r}\right)_{r=0} = \sin\varphi\ , \qquad \left(\frac{\partial R_2}{\partial \theta}\right)_{r=0} = 0$$

$$\left(\frac{\partial \tilde{\theta}_2}{\partial r}\right)_{r=0} = \frac{\cos\varphi}{R_c}\ , \qquad \left(\frac{\partial \tilde{\theta}_2}{\partial \theta}\right)_{r=0} = 0 \tag{5.4}$$

where R_s is the radius of the sphere.

Assuming $R_s = 1$ and using the relation $(\tilde{\theta}_2)_{r=0} = \frac{1}{2}\pi - \theta_c$, for coefficients in the pressure expansion, we have

$$(p_{00})_s^* = (p_0)_{r=0}$$

$$(p_{01})_s^* = \sin\varphi^* \left(\frac{\partial p_0}{\partial R_2}\right)_{r=0} + \cos\varphi^* \left(\frac{\partial p_0}{\partial \tilde{\theta}_2}\right)_{r=0}$$

$$(p_{01}')_s^* = \cos\varphi^* \left(\frac{\partial p_0}{\partial R_2}\right)_{r=0} - \sin\varphi^* \left(\frac{\partial p_0}{\partial \tilde{\theta}_2}\right)_{r=0}$$

$$(p_{\alpha 0})_s^* = - \left(\frac{\partial p_0}{\partial \tilde{\theta}_2}\right)_{r=0} \tag{5.5}$$

$$(p_{\alpha 1})_s^* = - \sin\varphi^* \left(\frac{\partial^2 p_0}{\partial \tilde{\theta}_2 \partial R_2}\right)_{r=0} - \cos\varphi^* \left(\frac{\partial^2 p_0}{\partial \tilde{\theta}_2^2}\right)_{r=0}$$

$$(p_{\dot{\alpha}0})_s^* = - \frac{\ell_1}{V_1}(p_{\alpha 0})_c^*\ , \qquad (p_{\dot{\alpha}1})_s^* = - \frac{\ell_1}{V_1}(p_{\alpha 1})_s^*$$

Equations (4.5) determine the relation between values of the coefficients of expansion in r on the line $\theta = \theta^*$ by approaching the line from both sides. For the pressure, in particular, we have

$$p_{00}^* = (p_{00})_s^*, \quad p_{01}^* = (p_{01})_s^*, \quad p_{\alpha 0}^* = (p_{\alpha 0})_s^*, \quad p_{\alpha 1}^* = (p_{\alpha 1})_s^* + \theta_{\alpha 0}\left[p_{01}'^*\right]$$

$$p_{\dot\alpha 0}^* = (p_{\dot\alpha 0})_s^*, \quad p_{\dot\alpha 1}^* = (p_{\dot\alpha 1})_s^* + \theta_{\dot\alpha 0}\left[p_{01}'^*\right] \tag{5.6}$$

The magnitude of the jump in $\left[p_{01}'^*\right]$ in these formulas, according to (3.7) is determined by the relation

$$\left[p_{01}'^*\right] = (p_{01}')_s^* + \tan \varphi^* (p_{01})_s^* \tag{5.7}$$

and the parameters $\theta_{\alpha 0}$ and $\theta_{\dot\alpha 0}'$ by the relations (4.4)

$$a_{00}'^* = 0, \qquad v_{0\theta 0}'^* = -\cos \varphi^* (v_{0\tilde\theta})_{r=0}$$

$$a_{\alpha 0}^* = -\left(\frac{\partial a_0}{\partial \tilde\theta_2}\right)_{r=0}, \qquad v_{\alpha\theta 0}^* = \sin \varphi^* \left(\frac{\partial v_{0\tilde\theta}}{\partial \tilde\theta_2}\right)_{r=0} \tag{5.8}$$

$$a_{\dot\alpha 0}^* = -\frac{\ell_1}{V_1} a_{\alpha 0}^*, \qquad v_{\dot\alpha\theta 0}^* = -\frac{\ell_1}{V_1} v_{\alpha\theta 0}^* + \sin \varphi^*$$

§6. Let us consider how to determine the aerodynamic moment M_2, acting on the oscillating body in supersonic flow. The moment is made up of two parts. The first part $(M_z)_s$ is created by the pressure force, acting on the spherical part of the body surface, the second is the pressure force acting on the conical part of the body surface. We introduce the pressure coefficient by the formula

$$C_{mz} = \frac{M_z}{\frac{1}{2}\rho_1 V_1^2 L\pi (L \tan \theta_c)^2} \tag{6.1}$$

where ρ_1 is the density in the undisturbed flow, and L is a characteristic length of the body (Fig. 1).

Then the derivatives of the moment coefficient with respect to α and β, where $\beta = \dot{\alpha}L/V_1$, can be represented for the spherical part in the form

$$(C_{mz\alpha})_s = 4\left[1 + \frac{2}{(\gamma-1)M_1^2}\right]\left(\frac{\ell}{L} - \frac{x_0}{L}\frac{1}{\cos^2\theta_c}\right)\left(\frac{x_0}{L}\right)^2 \frac{1}{\cos^2\theta_c} \times$$

$$\times \int_0^{\pi/2-\theta_c} \left[p_0 - (p_0)_{r=0}\right] \sin\tilde{\theta}\,\cos\tilde{\theta}\,d\tilde{\theta} \tag{6.2}$$

$$(G_{mz\beta})_s = -\left(\frac{\ell}{L} - \frac{x_0}{L}\frac{}{\cos^2\theta_c}\right)(C_{mz\alpha})_s$$

and for the conical part in the form

$$(G_{mz\alpha})_c = 2\left[1 + \frac{2}{(\gamma-1)M_1^2}\right]\left[\frac{p_{\alpha_0}^{o}}{\sin\theta_c\,\cos\theta_c} f_1\left(\frac{\ell}{L}, \frac{x_0}{L}, \theta_c\right) + \right.$$

$$\left. + \frac{p_{\alpha_1}^{o}}{\sin^2\theta_c} f_2\left(\frac{\ell}{L}, \frac{x_0}{L}, \theta_c\right)\right] \tag{6.3}$$

$$(C_{mz\beta})_c = 2\left[1 + \frac{2}{(\gamma-1)M_1^2}\right]\left[\frac{p_{\dot{\alpha}_0}^{o}}{\sin\theta_c \cos\theta_c} f_1\left(\frac{\ell}{L}, \frac{x_0}{L}, \theta_c\right) + \right.$$

$$\left. + \frac{p_{\dot{\alpha}_1}^{o}}{\sin^2\theta_c} f_2\left(\frac{\ell}{L}, \frac{x_0}{L}, \theta_c\right)\right] \tag{6.3 (Cont.)}$$

where

$$f_1\left(\frac{\ell}{L}, \frac{x_0}{L}, \theta_c\right) = \frac{1}{3}\left[1 - \left(\frac{x_0}{L}\right)^3\right] - \frac{1}{2}\frac{\ell}{L}\cos^2\theta_c\left[1 - \left(\frac{x_0}{L}\right)^2\right]$$

$$f_2\left(\frac{\ell}{L}, \frac{x_0}{L}, \theta_c\right) = \frac{1}{x_0/L}\left\{\frac{1}{4} + \frac{1}{12}\left(\frac{x_0}{L}\right)^4 - \frac{1}{3}\frac{x_0}{L} + \right. \tag{6.4}$$

$$\left. + \frac{\ell}{L}\cos^2\theta_c\left[\frac{1}{2}\frac{x_0}{L} - \frac{1}{3} - \frac{1}{6}\left(\frac{x_0}{L}\right)^3\right]\right\}$$

and the parameters $p_{\alpha 0}^{o}$, $p_{\alpha 1}^{o}$, $p_{\dot{\alpha} 0}^{o}$, $p_{\dot{\alpha} 1}^{o}$ are considered by expansion of the nondimensional pressure perturbation

$$\frac{p - p_0}{\rho_1 V_{max}^2} = \alpha\left(p_{\alpha_0}^{o} + \frac{r}{R_s} p_{\alpha 1}^{o}\right)\cos\psi + \beta\left(p_{\dot{\alpha}1}^{o} + \frac{r}{R_s} p_{\dot{\alpha}1}^{o}\right)\cos\psi \tag{6.5}$$

For convenience of comparison with the aerodynamic characteristics of conical bodies the geometrical parameters of the circumscribed cone are introduced in formulas (6.2) and (6.3) (see Fig. 1).

The method being suggested was used in the calculation for $M_1 = 4.0$. The necessary data on the parameters of the stationary stream line of the sphere are taken from paper [1]. In Figs. 3 and 4 are presented the functions

$$C_{mz\alpha} = (C_{mz\alpha})_s + (C_{mz\alpha})_c, \quad C_{mz\beta} = (C_{mz\beta})_s + (C_{mz\beta})_c \quad \text{for} \quad BC = R_s$$

$$(\beta = \dot{\alpha}L/V_1)$$

of the cone half angle θ_c and of the center of oscillations ℓ/L. A positive sign on $C_{mz\alpha}$ corresponds to statistical stability, and a positive sign on $C_{mz\beta}$ to damped oscillations. The graph in Fig. 5, where the value of $C_{mz\beta} \times 10^3$ is given, shows the contribution of the spherical part of the body to the aerodynamic moment. This contribution is small for large θ_c, but rapidly grows with decreased angle of attack, if only the center of oscillation is not too closely located to the center of the sphere.

§7. Now we will consider the flow over the sphere, performing plane angular oscillations with center 0 (see Fig. 2). The oscillations about a certain fixed angle-of-attack α^* are given by the law

$$\alpha = \alpha^* + \Delta\alpha_0 \cos \omega t \qquad (7.1)$$

In accord with Part 2 the perturbations caused in the flow by the angular oscillation of the sphere with center 0 are equivalent to translation with absolute velocity by the sphere with center 0_1. In the case of slow oscillations, if one is restricted to a linear approximation with respect to frequency, translations of the body cause the same perturbations in the gas flow as an imaginary change in both angle-of-attack and velocity of the undisturbed flow:

$$\Delta\alpha^*_\Phi = - \frac{\ell_1 \cos \alpha^*}{V_1} \dot{\alpha}, \quad \Delta V_{1\Phi} = \ell_1 \sin \alpha^* \dot{\alpha} \tag{7.2}$$

where ℓ_1 is the distance between points 0 and 0_1, positive at the rear position of the center of oscillations.

The moment coefficient, relative to the point 0, acting on the stationary streamline of the sphere under angle-of-attack α, is determined by the relation

$$C_{mz}(\alpha^*, M_1) = - \frac{\ell_1}{R_s} \sin \alpha^* C_x(M_1) \tag{7.3}$$

where the radius of the sphere is considered as the characteristic dimension, C_x is the drag coefficient of the sphere. Calculating the change in moment caused by a fictitious change of angle-of-attack and flow velocity (7.2) we obtain

$$\Delta C_{mz} = \frac{\dot{\alpha} R_s}{V_1} \left[C_x + \sin^2 \alpha^* \left(- C_x + M_1 \frac{\partial C_x}{\partial M_1} \right) \right] \left(\frac{\ell_1}{R_s} \right)^2$$

or taking into account (7.3),

$$C_{mz\alpha} = - \frac{\ell_1}{R_s} \cos \alpha^* C_x$$

$$\left(\beta = \frac{\dot{\alpha} R_s}{V_1} \right) \tag{7.4}$$

$$C_{mz\beta} = \left(\frac{\ell_1}{R_s} \right)^2 \left[C_x + \sin^2 \alpha^* \left(- C_x + M_1 \frac{\partial C_x}{\partial M_1} \right) \right]$$

In conclusion we note that the preceding derivatives of the moment coefficient are associated with the generally used coefficients of rotational derivatives C_{mz}^{α}, C_{mz}^{β}, $C_{mz}^{\Omega_z^o}$ in the following manner (see [2]):

$$C_{mz\alpha} = -G_{mz}^{\alpha}, \quad G_{mz\beta} = C_{mz}^{\beta} + C_{mz}^{\Omega_z^o} \qquad (\beta = \alpha^o L/V) \tag{7.5}$$

Submitted 10 Dec. 1960

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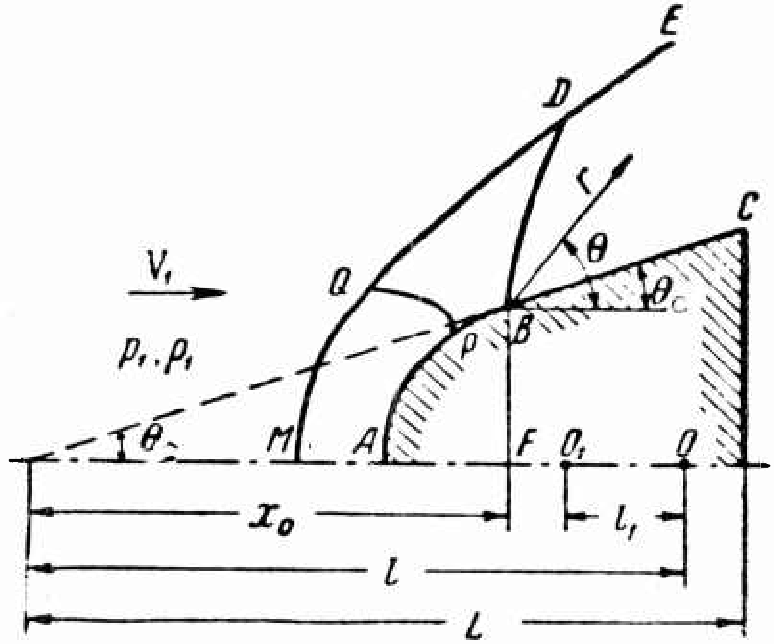


Fig. 1

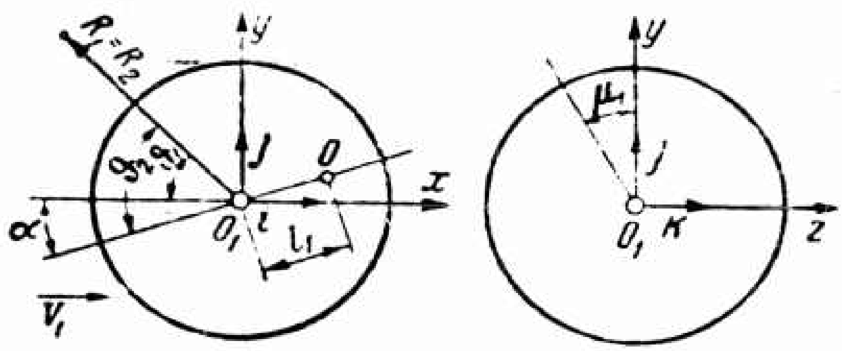
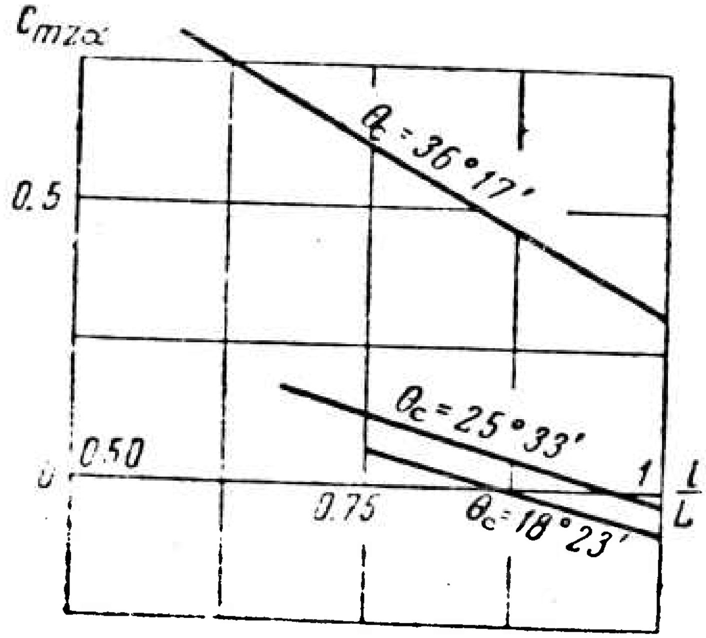


Fig. 2



$C_{mz\alpha}$
0.5
0
0.50
0.75
1
$\frac{l}{L}$
$\theta_c = 36°17'$
$\theta_c = 25°33'$
$\theta_c = 18°23'$

Fig. 3

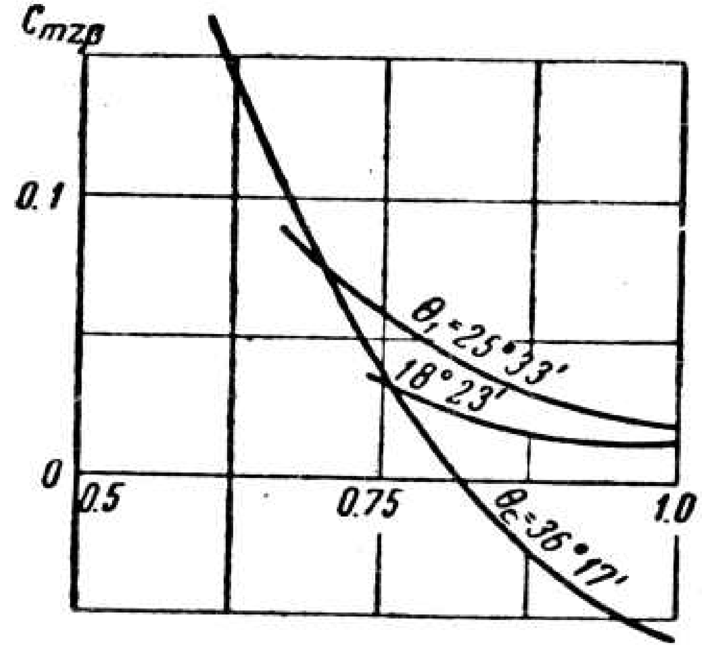
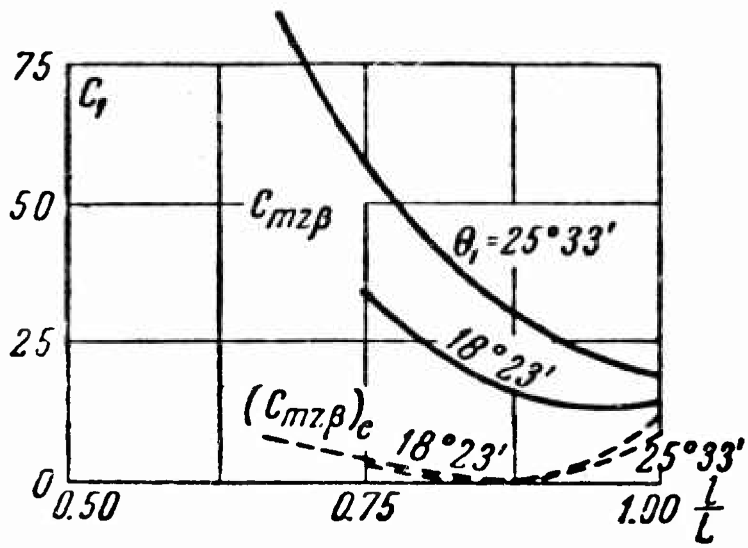


Fig. 4



75
50
25
0
C_1
$C_{mz\beta}$
$\theta_1=25°33'$
18°23'
$(C_{mz\beta})_c$
18°23'
25°33'
0.50
0.75
1.00
$\frac{l}{L}$

Fig. 5